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On strong and weak Bäcklund transformations

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Abstract. We show, on a set of examples containing the hierarchies of equations associated with the Zakharov and Shabat spectral problem, the modified Korteweg-de Vries-Burgers spectral problem and the chiral field equation, that one can easily prove the strength or weakness of a Bäcklund transformation for a whole hierarchy of equations by the use of the Darboux matrix approach and thus with just the knowledge of half a Bäcklund transformation.

Through a Bäcklund transformation (BT) one reduces the integration of a nonlinear partial differential equation (NPDE) to the solution of ordinary differential equations (ODE's), generally of low order. If these ODE's relate two different solutions $(q, \hat{q})$ of the same NPDE, we speak of an auto-BT. In the following we shall concentrate, for the sake of simplicity, only on this case. Once a solution q of the given NPDE is known, the BT is just an ODE for the field $\hat{q}$, whose coefficients are built up from the known solution q and its derivatives and its solution gives a *new* solution $\hat{q}$ of the given NPDE.

Many methods have been discovered and applied to the construction of BT's for NPDE's. Here we just mention the Clairin method (Lamb 1974, 1976), the geometrical approach (Pirani *et al* 1979) and the Darboux matrix technique (Levi and Ragnisco 1982, Levi *et al* 1982). All these methods are very important in view of the practical value represented by exact solutions for NPDE's and in the connection between these transformations and the remarkable properties associated with these NPDE's such as the existence of an infinite set of constants of the motion in involution, Hamiltonian structure, etc.

All the famous examples of BT's up to now considered can be divided into two classes: the weak BT and the strong BT. Here we adopt the McCarthy (1978) definition of weak and strong BT's, namely we call a BT strong if the transformation equations themselves already imply the NPDE. On the contrary a BT is weak if we must assume that the NPDE, for the field q, is satisfied in order to deduce from the transformation equations that the field $\hat{q}$ also satisfies the NPDE.

It is reasonable to ask oneself under what conditions a BT is strong or weak, how to recognise this property easily and what it implies.

In the following we shall show that the property of a BT being strong or weak is coded into the form of the Darboux matrix and in the spectral operator (SO) to which the NPDE is associated; thus this property applies to the whole hierarchy of NPDE's

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associated with the SO under study. This statement shall be shown generally for the Zakharov-Shabat 2×2 SO (Zakharov and Shabat 1972) and then, as further examples, we shall consider the modified Korteweg-de Vries-Burgers SO (Levi *et al* 1984) and the chiral field case (Zakharov and Mikhailov 1978).

Let us consider the 2×2 Zakharov-Shabat (ZS) SO:

$$\phi_x(x,t;\lambda)=\begin{pmatrix}\lambda & q(x,t)\\ r(x,t) & -\lambda\end{pmatrix}\phi(x,t;\lambda)=U(q,r;\lambda)\phi(x,t;\lambda). \tag{1}$$

By appropriate choice of the time evolution of the matrix wavefunction $\phi(x,t;\lambda)$

$$\phi_t(x,t;\lambda)=V(q,r;\lambda)\phi(x,t;\lambda), \tag{2}$$

the hierarchy of systems of NPDE's associated with (1) is obtained as compatibility of (1) and (2), i.e. through the condition that there exists a solution of the overdetermined system of equations (1) and (2) for ϕ

$$U_t-V_x+[U,V]=\begin{pmatrix}0 & X\\ Y & 0\end{pmatrix}=M=0. \tag{3}$$

A BT will be strong if $M=0$ can be derived just from the BT.

Let us consider the ZO SO corresponding to a different solution $(\hat{q},\hat{r})$ of the NPDE's (3), i.e.

$$\hat{\phi}_x(x,t;\lambda)=\begin{pmatrix}\lambda & \hat{q}\\ \hat{r} & -\lambda\end{pmatrix}\hat{\phi}(x,t;\lambda)=U(\hat{q},\hat{r};\lambda)\hat{\phi}(x,t;\lambda) \tag{4}$$

and its corresponding wavefunction time evolution

$$\hat{\phi}_t(x,t;\lambda)=V(\hat{q},\hat{r};\lambda)\hat{\phi}(x,t;\lambda) \tag{5}$$

such that $(\hat{q},\hat{r})$ satisfy the same NPDE's (3):

$$u_t-v_x+[u,v]=\begin{pmatrix}0 & \hat{X}\\ \hat{Y} & 0\end{pmatrix}=\hat{M}=0. \tag{6}$$

As was shown previously (Levi *et al* 1982), one can define an invertible gauge transformation D between ϕ and $\hat{\phi}$, often called a Darboux matrix, such that

$$\hat{\phi}=D(q,r;\hat{q},\hat{r};\lambda)\phi \tag{7}$$

which, in the simplest non-trivial case, corresponding to the addition of one pole in the spectrum of (1), can be written as

$$D=D_0(x,t)+\lambda D_1(x,t).$$

For the 2×2 ZS SO case, D reads

$$D=\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}(p_1^+-p_1^-)[\lambda+\frac{1}{2}J(x)]+p_0^+-p_0^-; & -\frac{1}{2}(p_1^++p_1^-)\hat{q}+\frac{1}{2}(p_1^+-p_1^-)q\\ \frac{1}{2}(p_1^+-p_1^-)\hat{r}-\frac{1}{2}(p_1^++p_1^-)r; & (p_1^++p_1^-)[\lambda-\frac{1}{2}J(x)]+p_0^++p_0^-\end{pmatrix}$$

where $\det D=[\lambda(p_1^+-p_1^-)+(p_0^+-p_0^-)][\lambda(p_1^++p_1^-)+(p_0^++p_0^-)]$, $p_0^\pm, p_1^\pm$ being arbitrary complex scalar constants and

$$J(x)=\int_x^\infty \mathrm{d}x'[q(x')r(x')-\hat{q}(x')\hat{r}(x')].$$

As is well known (Levi and Ragnisco 1982) by requiring the compatibility of (1), (2), (4), (5) and (7), we get

$$D_x = U(\hat{q}, \hat{r}; \lambda)D - DU(q, r; \lambda) \tag{8}$$

$$D_t = V(\hat{q}, \hat{r}; \lambda)D - DV(q, r; \lambda), \tag{9}$$

which are the BT's written in compact form; the explicit form of (8) in terms of the potentials (q, r), $(\hat{q}, \hat{r})$ is

$$\begin{aligned} &p_1^+[\hat{q}_x - q_x - (\hat{q}+q)J(x)] + p_1^-[\hat{q}_x + q_x - (\hat{q}-q)J(x)] + 2p_0^+(\hat{q}-q) + 2p_0^-(\hat{q}+q) = 0, \\ &p_1^+[\hat{r}_x - r_x - (\hat{r}+r)J(x)] - p_1^-[\hat{r}_x + r_x - (\hat{r}-r)J(x)] - 2p_0^+(\hat{r}-r) + 2p_0^-(\hat{r}+r) = 0. \end{aligned} \tag{10}$$

In order to verify whether this BT is strong or weak we can check the integrability condition between equations (8) and (9); this implies

$$\hat{M} = DMD^{-1} \tag{11}$$

i.e. in the 2×2 ZS SO case

$$\det D\begin{pmatrix} 0 & \hat{X} \\ \hat{Y} & 0 \end{pmatrix} = \begin{pmatrix} bdY - acX; & a^2X - b^2Y \\ d^2Y - c^2X; & acX - bdY \end{pmatrix}. \tag{12}$$

From formula (12), equating the terms corresponding to equal powers of λ, we derive immediately that for all the NPDE's associated with the 2×2 ZS SO, the BT (10) is strong. Thus we see immediately that all the information on the weakness of strength of the BT is coded into the integrability condition (12).

It is worthwhile noticing that we should expect the 'adding one solition' BT for the ZS SO for matrices of rank n, with $n>2$, to be weak, as this is the case of the SU(n) chiral field equation which we shall consider at the end.

As a further example, let us consider the following spectral operator introduced a few years ago by one of the authors (Levi 1981)

$$\phi_x(x, t; \lambda) = \begin{pmatrix} -u(x, t) & \lambda u(x, t) \\ -\lambda & \lambda^2 + v(x, t) \end{pmatrix} \phi(x, t; \lambda) \tag{13}$$

where, as in the previous case, ϕ is a matrix of rank 2, $u(x, t)$ and $v(x, t)$ are two scalar asymptotically vanishing potentials and λ is the spectral parameter. In subsequent work (Levi *et al* 1984) we have been able to transform it into the ZS SO (1), to obtain directly its N-soliton solutions and BT's. From Levi *et al* (1984) we recall the simplest non-trivial equations associated with (13)

$$u_t = (u_x + 2uv - v^2)_x \qquad v_t = (-v_x - 2uv + v^2)_x \tag{14}$$

$$\begin{aligned} u_t &= (u_{xx} + 3u_xv - 3uu_x + u^3 + 3uv^2 - 6u^2v)_x \\ v_t &= (v_{xx} + 3v_xu - 3vv_x + v^3 + 3u^2v - 6uv^2)_x. \end{aligned} \tag{15}$$

Equations (14) and (15) admit many reductions to a single NPDE as, for example, the Burgers hierarchy ($v=0$ or $u=0$) (Levi *et al* 1983), the modified KdV hierarchy ($u=v$) and a new nonlinear Schrödinger hierarchy ($u=v$) (Levi *et al* 1984, Nijhoff *et al* 1983). In this case (3) and (6) imply

$$M = \begin{pmatrix} -X, & \lambda X \\ 0, & Y \end{pmatrix}; \qquad \hat{M} = \begin{pmatrix} -\hat{X}, & \lambda\hat{X} \\ 0, & \hat{Y} \end{pmatrix},$$

and thus equation (11) becomes

$$\det D\begin{pmatrix} -\hat{X}, & \lambda\hat{X} \\ 0, & \hat{Y} \end{pmatrix} = \begin{pmatrix} -a(d+\lambda c)X - bcY; & a[(b+\lambda a)X+bY] \\ -c[(d+\lambda c)X+dY]; & c(b+\lambda a)X+daY \end{pmatrix}. \tag{16}$$

Among the various Darboux matrices considered (Levi *et al* 1984) we consider here, as an example, the following one

$$D=\begin{pmatrix} \lambda^2\alpha+\gamma-\beta K(x)\hat{u}; & \lambda[\beta K(x)\hat{u}-\alpha u] \\ \lambda[\alpha-\beta K(x)]; & \lambda^2\beta K(x)+\gamma+\beta K(x)v-\alpha[\hat{v}+J(x)] \end{pmatrix} \tag{17}$$

where α, β, γ are arbitrary, *a priori* complex, constants and $K(x)$ and $J(x)$ are such that $K_x(x)=K(x)(\hat{v}-v+u-\hat{u})$, $J_x(x)=\hat{v}\hat{u}-vu$. From equation (16) we get

$$Y=-(1+\lambda c/d)X;\quad \hat{X}=-Y;\quad \hat{X}=ad/X;\quad \hat{Y}=-X$$

which imply $X=0$ and consequently $\hat{X}=0$, $Y=0$, $\hat{Y}=0$ as, from (17), $a+d+\lambda c\neq 0$.

To end we consider the case of the two-dimensional SU(n) chiral field equation, given by

$$(g^+g_x)_t(g^+g_t)_x=0 \tag{18}$$

where the $n\times n$ matrix g belong to SU(n) and by + we denote Hermitian conjugation. As was shown by Zakharov and Mikhailov (1978) equation (18) can be cast as the compatibility condition between the following generalised Lax pair:

$$\phi_x=\frac{1}{1-\lambda}(g^+g_x)\phi=U\phi, \qquad \phi_t=\frac{1}{1+\lambda}(g^+g_t)\phi=V\phi,$$

where λ is a spectral parameter and ϕ is a matrix function of rank n belonging to SU(n). The usual BT (Ogielski *et al* 1980) can be obtained from a Darboux matrix

$$D=(2-\lambda)+d\hat{g}^+g \tag{19}$$

where d is an arbitrary complex constant and we must have

$$d\hat{g}^+g+d^*g^+\hat{g}=d+d^*, \tag{20}$$

D^{-1} is given by

$$D^{-1}=[2-\lambda+d^*g^+\hat{g}]/[(2-\lambda)^2+(2-\lambda)\alpha+|d|^2],$$

and thus equation (11) implies

$$\hat{M}=M \tag{21}$$

$$(d+d^*)\hat{M}=dM\hat{g}^+g+d^*g^+\hat{g}M \tag{22}$$

$$\hat{M}=g^+\hat{g}M\hat{g}^+g. \tag{23}$$

By taking into account equation (20) one can easily prove that the condition (22) is identically satisfied if equations (21) and (23) are. Then, by the position $g=h^+$, $\hat{g}=\hat{h}^+$, (23) is identical with (21), so that we can infer that only if $M=0$, i.e. only if the BT starts from a solution of our NPDE(18), will the resulting solution of the BT give rise to a new solution of our NPDE(18). So this proves that the BT associated to the Darboux matrix (19) is weak.

The same result could be proved, in a completely analogous way, for the four-dimensional self-dual Yang-Mills theory (Bruschi *et al* 1982, Popowicz 1983).

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